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THE METHOD OF PARAMETER FUNCTIONALIZATION IN THE HOPF BIFURCATION PROBLEM

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1. INTRODUCTION

QUITE a number of investigations are devoted to the problem of generation of small periodic oscillations of the system of ordinary differential equations

$$\frac{dx}{dt} = F(\lambda, x) \qquad (F(\lambda, 0) \equiv 0) \tag{1}$$

depending on a parameter (for an extensive bibliography see [1]). The greater part of these investigations exploit information about not only the linear terms of the right-hand member of the system (1), the linear operator $F_x'(\lambda, 0)$, but also information about the terms of higher orders in the Taylor expansion in x of the function $F(\lambda, x)$. With the help of such information one can answer the questions on the number of arising self-oscillations, their stability, their dependence on a parameter, etc. Proofs of corresponding assertions use, as a rule, an analytic technique such as varied forms of the implicit function theorem, the theory of invariant manifolds, and the like.

At the same time, in problems arising during the study of complicated physical, technological, etc. processes, often the only rather complete information available is that concerning the linear terms of the right-hand member of the system (1). In such circumstances it is very difficult to apply an analytic technique for studying the problem of generation of self-oscillations.

Yet as things turn out [2], in some cases the very fact of generation of small self-oscillations can be picked out of information about the linear terms of the right-hand member of the system (1). Of course, under lack of information about the high order terms in the Taylor expansion in x of $F(\lambda, x)$ one can say next to nothing about properties of arising self-oscillations.

This article contains a topological proof of the Hopf theorem. In this proof the method of parameter functionalization [3], introduced by Krasnosel'skii in another situation, has much significance. Employment of topological considerations makes it possible to throw aside usual assumptions of differentiability of the right-hand member of the system (1). This provides an opportunity to investigate systems (1), the right-hand side of which contains, for example, hysteresis-type or relay-type nonlinearities. The proof presented below goes through without changes in the case of functional differential equations with lagging arguments [4]. We do not present here the exact formulations of such assertions because our purpose is the demonstration of the method. In [5, 6] the method of parameter functionalization has been applied to the investigation of bifurcation of long-periodic solutions of differential equations and mappings.

2. THE MAIN RESULT

From now on we shall assume that the parameter λ in the system (1) is real.

We shall say that for $\lambda = \lambda_0$ generation of small periodic solutions of the system (1) with periods close to T_0 takes place, if for every $\varepsilon > 0$ there exists a λ_ε in the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ for which the system (1) has a nonzero T_ε-periodic solution $x_\varepsilon(t)$ ($|T_\varepsilon - T_0| < \varepsilon$) lying in the ε-neighborhood of zero

$$\|x_\varepsilon(t)\| < \varepsilon \qquad (-\infty < t < \infty).$$

Let us suppose that the right-hand member of the system (1) depends continuously on λ and x in a neighborhood of the point $\{\lambda_0, 0\} \in R^1 \times R^n$ and admits a representation of the form

$$F(\lambda, x) = A(\lambda)x + a(\lambda, x), \tag{2}$$

where $A(\lambda)$ is a matrix and where the remainder term $a(\lambda, x)$ satisfies the condition

$$\lim_{\|x\|\to 0} \frac{\|a(\lambda, x)\|}{\|x\|} = 0 \tag{3}$$

uniformly with respect to λ. Then as is easy to see the matrix $A(\lambda)$ depends continuously on λ in a neighborhood of λ_0 and the function $a(\lambda, x)$ depends continuously on λ and x in a neighborhood of the point $\{\lambda_0, 0\}$.

Let the matrix $A(\lambda_0)$ have the purely imaginary eigenvalue $i\omega_0$ ($\omega_0 \neq 0$) which is simple. Then as is known for λ close to λ_0 the matrix $A(\lambda)$ has a unique eigenvalue $\mu(\lambda)$, that is close to $i\omega_0$. Moreover, the function $\mu(\lambda)$ depends continuously on λ in a neighborhood of λ_0.

THEOREM. Let the matrix $A(\lambda_0)$ have no eigenvalues of the form $0, \pm 2i\omega_0, \pm 3i\omega_0, \ldots$ and let the real part $\mathrm{Re}\,\mu(\lambda)$ of the eigenvalue $\mu(\lambda)$ takes values of opposite signs in every neighborhood of λ_0.

Then for $\lambda = \lambda_0$ generation of small periodic solutions of the system (1) with periods close to $2\pi/\omega_0$ takes place.

The idea of the proof of the theorem (the method of parameter functionalization) is a fairly simple one. First, we reduce the problem of the existence of periodic solutions of the system (1) to the problem of the existence of solutions of some operator equation

$$x = U(\nu, x)$$

that depends on the two-dimensional parameter $\nu = \{T, \lambda\}$ where T is an unknown period of a periodic solution to be found. Afterwards we construct such a sequence of functionals $\{\nu_n(x)\}$ that for each n the rotation of the vector field $x - U[\nu_n(x), x]$ on the boundary of some region Ω_n ($0 \notin \Omega_n$) is not equal to zero. Then the operator $U[\nu_n(x), x]$ has at least one nonzero fixed point x_n in the region Ω_n. Evidently, this point is a nonzero solution of the equation

$$x = U(\nu_n, x)$$

where $\nu_n = \{T_n, \lambda_n\} = \nu_n(x_n)$. It follows from this that for $\lambda = \lambda_n$ the system (1) has a nonzero periodic solution $x_n(t)$ with the period T_n. The functionals $\nu_n(x)$ and the regions Ω_n can be constructed in such a way that $\Omega_n \to 0$, $\nu_n(\Omega_n) \to \{2\pi/\omega_0, \lambda_0\}$. Hence $\lambda_n \to \lambda_0$, $T_n \to 2\pi/\omega_0$ and the amplitudes of the corresponding periodic solutions $x_n(t)$ tend to zero.

3. PRELIMINARIES

Consider in a real Banach space E a linear bounded operator $V(\nu)$ which depends on a parameter ν from a Banach space N. Let the operator $V(\nu)\,x$ be completely continuous as an operator from $\mathrm{N} \times E$ into E, then the principal spectral properties of $V(\nu)$ are the same as if $V(\nu)$ depended continuously on ν with respect to the norm of the operators. This is a simple but important fact, since linear operators naturally arising in the theory of differential equations in some cases do not possess the property of continuity on ν with respect to the norm of the operators but do possess the property of complete continuity on ν and x. To a far greater extent the same is valid for operators arising in the theory of differential equations with lagging arguments.

Let us describe some spectral properties of the operator $V(\nu)$. For details and proofs, see [7].

The basic spectral property of the operator $V(\nu)$ contained in the fact that the spectrum of $V(\nu)$ depends continuously on ν in the Hausdorff metric.

If μ_0 is an isolated eigenvalue of the operator $V(\nu_0)$ then the Riesz's formula

$$P(\nu) = \mathrm{Re}\left\{\frac{1}{2\pi i}\int_\gamma [\lambda I - V(\nu)]^{-1}\mathrm{d}\lambda\right\}$$

where γ is a closed curve in the complex plane with μ_0 in its interior and the rest of the spectrum of $V(\nu_0)$ in its exterior, defines for $\nu = \nu_0$ the real projector onto the generalized eigenspace of μ_0. In virtue of continuity of the spectrum of $V(\nu)$ for every ν close to ν_0 the same formula defines some projector $P(\nu)$ that commutes with $V(\nu)$. Hence $P(\nu)$ is projected onto some $V(\nu)$-invariant subspace of the space E.

Rewrite Riesz's formula in the equivalent form

$$P(\nu) = \mathrm{Re}\left\{\frac{1}{2\pi i}\int_\gamma \frac{1}{\lambda}[\lambda I - V(\nu)]^{-1}V(\nu)\,\mathrm{d}\lambda\right\}.$$

From this one can easily see that $P(\nu)x$, the same as $V(\nu)x$, depend completely continuously on ν and x.

From the above properties of the projector $P(\nu)$ and of the spectrum of $V(\nu)$ one can derive in a general way the other spectral properties of $V(\nu)$. For example, if μ_0 is a simple eigenvalue of the operator $V(\nu_0)$ then the ν close to ν_0 the linear operator $V(\nu)$ has a unique eigenvalue $\mu(\nu)$, that is close to μ_0. This eigenvalue is simple, it and its eigenvector depend continuously on the parameter. If the eigenvalue μ_0 is not a simple one then for values of ν close to ν_0 the operator $V(\nu)$ has not one but, generally speaking, a group of eigenvalues, that are close to μ_0. The sum of the multiplicities of these eigenvalues is equal to the multiplicity of μ_0. Behaviour of these eigenvalues is highly complicated. For example, they may join, bifurcate, change the order, etc.

We are most interested in the case when μ_0 is a real eigenvalue of the order and multiplicity 2. In this case we shall say that μ_0 is the eigenvalue of simple structure if in some neighborhood of ν_0 there exist continuous linearly independent real vectors $g(\nu)$ and $h(\nu)$ and a function $\mu(\nu)$ (complex, generally speaking) such that

$$V(\nu)\{g(\nu) + ih(\nu)\} = \mu(\nu)\{g(\nu) + ih(\nu)\}. \tag{4}$$

In this case we shall call the function $\mu(\nu)$ the continuous branch of eigenvalues passing for $\nu = \nu_0$ through the eigenvalue of simple structure μ_0.

Let us present an example. Let $V(\nu_0)$ have the simple eigenvalue $2\pi i$ and have no eigenvalues of the form $0, \pm 4\pi i, \pm 6\pi i, \ldots$ Then the eigenvalue 1 of the linear operator $e^{V(\nu_0)}$ has the order and multiplicity 2. This eigenvalue is of simple structure.

Let us consider an operator $U(\nu, x)$ which is defined on a neighborhood of a point $\{\nu_0, 0\} \in R^2 \times E$ and takes values in a real Banach space E. Let the operator $U(\nu, x)$ be completely continuous with respect to both variables and admit a representation of the form

$$U(\nu, x) = V(\nu)x + v(\nu, x),$$

where $V(\nu)$ is a linear bounded operator and where the remainder term $v(\nu, x)$ satisfies the condition

$$\lim_{\|x\| \to 0} \frac{\|v(\nu, x)\|}{\|x\|} = 0$$

uniformly with respect to ν from a neighborhood of ν_0. We should like to note that under such conditions both the operators, $V(\nu)x$ and $v(\nu, x)$ depend completely continuously on ν and x (see, for example, [3]).

LEMMA. Let 1 be the eigenvalue of the order and multiplicity 2 of the linear operator $V(\nu_0)$ and let this eigenvalue be of simple structure. Let in R^2 a sequence of Jordan curves $\{L_n\}$ converging to ν_0 exist. Let the rotation of the vector field $1 - \mu(\nu)$, where $\mu(\nu)$ is the continuous branch of eigenvalues passing through the eigenvalue 1, be defined and not equal to zero on each curve L_n.

Then there exist $\nu_n \to \nu_0$ and $x_n \to 0$ ($x_n \neq 0$) such that $x_n = U(\nu_n, x_n)$.

This lemma plays an important role in the proof of the theorem. A generalization of this lemma for the case of n-dimensional parameter see in [4]. The proof of the lemma will be presented in Section 5.

4. PROOF OF THE THEOREM

Let $C([0, \tau]; R^n)$ denote the Banach space of continuous R^n-valued functions defined on the interval $[0, \tau]$, where $\tau = 2\pi/\omega_0 + 1$, with the topology of uniform convergence. Consider in $C([0, \tau]; R^n)$ an operator of the form

$$U(T, \lambda; x)(t) = e^{tA(\lambda)}x(T) + \int_0^t e^{(t-s)A(\lambda)}\, a[\lambda, x(s)]\, ds$$

which depends on two real parameters T and λ. Here $A(\lambda)$ is the linear term and $a(\lambda, x)$ is the remainder term (see (2)) of the right-hand member of the system (1).

Direct verification shows that $x \in C([0, \tau]; R^n)$ is a fixed point of the operator $U(T, \lambda; \cdot)$ if and only if $x(t)$ is a solution of the system (1) that satisfies the condition $x(0) = x(T)$. This condition means that $x(t)$ is a T-periodic solution of the system (1). Hence the theorem would be proved if there existed $T_n \to 2\pi/\omega_0$ and $\lambda_n \to \lambda_0$ such that the operators $U(T_n, \lambda_n; \cdot)$ had nonzero fixed points x_n converging to zero. To prove it we shall use the lemma.

Let us verify if the operator $U(T, \lambda; x)$ satisfies the lemma's conditions.

As is easy to see, the operator $U(T, \lambda; x)$ is completely continuous with respect to the variables T, λ and x. Represent it in the form of a sum

$$U(T, \lambda; x) = V(T, \lambda)x + v(T, \lambda; x),$$

where

$$V(T, \lambda)x(t) = \mathrm{e}^{tA(\lambda)}\, x(T),$$

$$v(T, \lambda; x)(t) = \int_0^t \mathrm{e}^{(t-s)A(\lambda)}\, a[\lambda, x(s)]\, \mathrm{d}s.$$

Because of the assumption (3), $a(\lambda, x)$ is the function of higher order of smallness than $\|x\|$ uniformly with respect to λ from a neighborhood of λ_0. Consequently the nonlinear operator $v(T, \lambda; x)$ also has higher order of smallness than $\|x\|$ uniformly with respect to T from the interval $[0, \tau]$ and λ from a neighborhood of λ_0.

Now verify the lemma's conditions relating to the linear operator $V(T, \lambda)$. First, let us clarify the spectral properties of $V(T, \lambda)$. Since for each T and λ the linear operator $V(T, \lambda)$ is completely continuous then by virtue of the Reisz–Schauder theory all the points of its spectrum (possibly, except the point 0) are eigenvalues of a finite multiplicity. Let κ be a nonzero eigenvalue of the operator $V(T, \lambda)$ and let x be the corresponding to κ eigenvector. This means that $\kappa x = V(T, \lambda)x$ or what is the same

$$\kappa x(t) = \mathrm{e}^{tA(\lambda)}\, x(T) \qquad (x(t) \not\equiv 0).$$

From this equation we conclude

$$\kappa x(T) = \mathrm{e}^{TA(\lambda)} x(T) \qquad (x(T) \neq 0).$$

Hence $\kappa \neq 0$ is an eigenvalue of the operator $V(T, \lambda)$ (and $x(t)$ is the corresponding eigenvector) if and only if κ is an eigenvalue of the operator $\mathrm{e}^{TA(\lambda)}$ (and $x(T)$ is the corresponding eigenvector). From this one can see that the order of the eigenvalue $\kappa \neq 0$ of the operator $V(T, \lambda)$ is the same as that of $\mathrm{e}^{TA(\lambda)}$. Likewise it can be seen that the multiplicity of the eigenvalue $\kappa \neq 0$ of the operator $V(T, \lambda)$ is the same as that of $\mathrm{e}^{TA(\lambda)}$.

Let us show that 1 is the eigenvalue of the operator $\mathrm{e}^{T_0 A(\lambda_0)}$, where $T_0 = 2\pi/\omega_0$, of the order and multiplicity 2. As is known (see, for example, [8]) 1 is the eigenvalue of the operator $\mathrm{e}^{T_0 A(\lambda_0)}$ if and only if there exists an eigenvalue μ of the operator $A(\lambda_0)$ such that

$$\mathrm{e}^{T_0 \mu} = 1. \tag{5}$$

Moreover, the order and multiplicity of the eigenvalue 1 are equal to the sum of the orders and multiplicities respectively of all the eigenvalues of the operator $A(\lambda_0)$ which satisfy (5). From this and from the equality $T_0 = 2\pi/\omega_0$ we get

$$\mu = \frac{2\pi k i}{T_0} = ik\omega_0 \qquad (k = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

In virtue of the various theorem's condition the operator $A(\lambda_0)$ has exactly two (simple) eigenvalues which satisfy (6). They are $i\omega_0$ and $-i\omega_0$. Thus 1 is the eigenvalue of the operator $\mathrm{e}^{T_0 A(\lambda_0)}$, and also of the operator $V(T_0, \lambda_0)$, of the order and multiplicity 2.

Let us show that 1 is the eigenvalue of simple structure. Let $\mu(\lambda)$ be the continuous branch of eigenvalues of the operator $A(\lambda)$ passing for $\lambda = \lambda_0$ through the eigenvalue $i\omega_0$, and let $z(\lambda) = g(\lambda) + ih(\lambda)$ be the normed continuous branch of eigenvectors of the operator $A(\lambda)$ corresponding to $\mu(\lambda)$. Then

$$A(\lambda)\{g(\lambda) + ih(\lambda)\} = \mu(\lambda)\{g(\lambda) + ih(\lambda)\}.$$

Rewrite this complex equality for $\lambda = \lambda_0$ in the form of two real equalities

$$A(\lambda_0)g(\lambda_0) = -\omega_0 h(\lambda_0), \qquad A(\lambda_0)h(\lambda_0) = \omega_0 g(\lambda_0).$$

From the latter equalities one can easily see that the vectors $g(\lambda_0)$ and $h(\lambda_0)$ are linearly independent. Consequently the vectors $g(\lambda)$ and $h(\lambda)$ are linearly independent for all the values of λ close to λ_0.

Let us set

$$\begin{aligned} \kappa(T, \lambda) &= e^{T\mu(\lambda)}, \\ g(\lambda; t) &= e^{tA(\lambda)}\, g(\lambda), \\ h(\lambda; t) &= e^{tA(\lambda)}\, h(\lambda). \end{aligned} \tag{7}$$

Then

$$e^{tA(\lambda)}\{g(\lambda; T) + ih(\lambda; T)\} = \kappa(T, \lambda)\{g(\lambda; t) + ih(\lambda; t)\}.$$

Hence by definition of the operator $V(T, \lambda)$ the eigenvalue $1 = \kappa(T_0, \lambda_0)$ is of simple structure and $\kappa(T, \lambda)$ is the continuous branch of eigenvalues of the linear operator $V(T, \lambda)$ passing for $T = T_0 = 2\pi/\omega_0$, $\lambda = \lambda_0$ through the eigenvalue 1.

To complete the verification of the lemma's conditions it remains only to construct required Jordan curves. Denote the rotation of a vector field Φ on the boundary L of some region by $\gamma(\Phi, L)$.

Let us represent the function $\kappa(T, \lambda)$ in the following form (see (7))

$$\kappa(T, \lambda) = 1 + \{T\mu(\lambda) - 2\pi i\} + o(T\mu(\lambda) - 2\pi i). \tag{8}$$

Note now that if for some Jordan curves $\{L_n\}$ converging to the point $\{T_0, \lambda_0\}$ the non-equalities

$$\gamma(2\pi i - T\mu(\lambda), L_n) \neq 0 \tag{9}$$

be fulfilled, then in virtue of Rouche's theorem (see [8]) and the representation (8), for each sufficiently large n the following relations should be fulfilled

$$\gamma(1 - \kappa(T, \lambda), L_n) = \gamma(2\pi i - T\mu(\lambda), L_n) \neq 0. \tag{10}$$

Therefore, it is sufficient to construct such curves $\{L_n\}$ that the non-equalities (9) would be fulfilled.

By virtue of the theorem's condition the function $\operatorname{Re}\mu(\lambda)$ changes the sign in every neighborhood of λ_0. Hence there exist $\lambda_n^-, \lambda_n^+ \to \lambda_0$ such that

$$\operatorname{Re}\mu(\lambda_n^-) < 0, \qquad \operatorname{Re}\mu(\lambda_n^+) > 0.$$

Denote $\max|\omega_0 - \operatorname{Im}\mu(\lambda)|$ for λ from the interval $[\lambda_n^-, \lambda_n^+]$ by m_n and set

$$T_n^- = \frac{2\pi}{\omega_0 + m_n + 1/n}, \qquad T_n^+ = \frac{2\pi}{\omega_0 - m_n - 1/n}.$$

Let us consider the Jordan curve L_n that is the boundary of the rectangle $T \in [T_n^-, T_n^+]$, $\lambda \in [\lambda_n^-, \lambda_n^+]$. Since $\lambda_n^-, \lambda_n^+ \to \lambda_0$ and $T_n^-, T_n^+ \to T_o = 2\pi/\omega_0$ then L_n tends to the point $\{T_0, \lambda_0\}$. Besides, one can easily see that

$$\begin{aligned}
&\mathrm{Re}\{2\pi i - T\mu(\lambda) > 0 \qquad && \text{for } T \in [T_n^-, T_n^+], \lambda = \lambda_n^-;\\
&\mathrm{Re}\{2\pi i - T\mu(\lambda) < 0 && \text{for } T \in [T_n^-, T_n^+], \lambda = \lambda_n^+;\\
&\mathrm{Im}\{2\pi i - T\mu(\lambda) > 0 && \text{for } T = T_n^-, \lambda \in [\lambda_n^-, \lambda_n^+];\\
&\mathrm{Im}\{2\pi i - T\mu(\lambda) < 0 && \text{for } T = T_n^+, \lambda \in [\lambda_n^-, \lambda_n^+].
\end{aligned}$$

These inequalities show that the vector field $2\pi i - T\mu(\lambda)$ does not vanish on L_n and that on opposite sides of the rectangle L_n vectors of the vector field $2\pi i - T\mu(\lambda)$ are not identically directed. Hence the relations (10) are valid (see, for example, [8]).

So, the operator $U(T, \lambda; x)$ satisfies all the conditions of the lemma. Therefore, there exist $T_n \to 2\pi/\omega_0$, $\lambda_n \to \lambda_0$ and nonzero fixed points x_n of the operators $U(T_n, \lambda_n; \cdot)$ such that $x_n \to 0$. The functions $x_n(t)$ are the periodic solutions to be found.

The theorem is proved.

5. PROOF OF THE LEMMA

We shall prove the lemma in three steps. First, we shall obtain some auxiliary properties of the operator $U(\nu, x)$. Then we shall get a number of estimates of the vector field $\Phi_n(x) = x - U[\nu_n(x), x]$, where $\nu_n(x)$ is a specially constructed nonlinear functional. Finally, we shall show that for each sufficiently large n the vector field $\Phi_n(x)$ has at least one singular point $x_n \neq 0$ (i.e. $\Phi_n(x_n) = 0$) and, what is more, $x_n \to 0$ and $\nu_n(x_n) \to \nu_0$. Denoting $\nu_n(x_n)$ by ν_n we shall derive from this what is required: $x_n = U(\nu_n, x_n)$, where $x_n \to 0$ ($x_n \neq 0$) and $\nu_n \to \nu_0$.

Step 1. Let P denote the spectral projector of the operator $V(\nu_0)$ onto the generalized eigenspace of the eigenvalue 1. As mentioned in Section 3, P commutes with $V(\nu_0)$, however one can easily see that generally speaking P does not commute with $V(\nu)$ for $\nu \neq \nu_0$. Yet it turns out that without loss of generality one can assume that

$$PV(\nu) \equiv V(\nu)P. \tag{11}$$

Moreover, if $\mu(\nu)$ is a continuous branch of eigenvalues of the operator $V(\nu)$ passing for $\nu = \nu_0$ through 1 then without loss of generality one can assume that in (4) the vectors $g(\nu)$ and $h(\nu)$ do not depend on ν:

$$V(\nu)\{g + ih\} \equiv \mu(\nu)\{g + ih\}. \tag{12}$$

Let us prove it. Choose in the complex plane a closed curve γ interior of which contains 1 and does not contain the rest of the spectrum of $V(\nu_0)$. Then for all the values of ν sufficiently close to ν_0 Riesz's formula defines the projector $P(\nu)$ ($P(\nu_0) = P$) that commutes with $V(\nu)$ (see Section 3).

In virtue of the lemma's condition the multiplicity of the eigenvalue 1 of the operator $V(\nu_0)$ is equal to 2. Therefore, the vectors $g = g(\nu_0)$ and $h = h(\nu_0)$ (see (4)) form a basis in the two-dimensional subspace PE. Denote the linear operator defined on PE that transforms the vectors g and h in the vectors $g(\nu)$ and $h(\nu)$ respectively by $H(\nu)$. Set

$$Q(\nu) = [I - P(\nu)][I - P] + H(\nu)P.$$

It is easy to see that $Q(\nu_0) = I$. Hence the operator $Q(\nu)$ for $\nu = \nu_0$ has the bounded inverse operator. Let us show that the operator $Q(\nu)$ has the bounded inverse operator defined on E for all the values of ν sufficiently close to ν_0. To prove this let us note that for every ν the operator $Q(\nu) - I$ is completely continuous. Therefore, in virtue of the Riesz–Schauder theory $Q(\nu)$ has the bounded inverse operator if and only if 0 is not its eigenvalue. Supposing the contrary one can find x_n ($\|x_n\| = 1$) and $\nu_n \to \nu_0$ such that $Q(\nu_n)x_n = 0$ or, what is the same,

$$[I - P(\nu_n)][I - P]x_n + H(\nu_n)Px_n = 0. \tag{13}$$

Rewrite this equality in the following form

$$x_n = P(\nu_n)x_n - P(\nu_n)Px_n + Px_n - H(\nu_n)Px_n.$$

Since the linear operator $P(\nu)x$ is completely continuous with respect to ν and x (see Section 3) then from the latter equality compactness of the sequence $\{x_n\}$ follows. Therefore without loss of generality we can assume that $x_n \to x_*$ where $\|x_*\| = 1$. Taking the limit in (13) we obtain

$$x_* = (I - P)(I - P)x_* + Px_* = 0.$$

We have come to the contradiction.

Thus, for ν close to ν_0 there exists the linear bounded operator $Q^{-1}(\nu)$ defined on E. The linear operators $P(\nu)$ and $H(\nu)$ are strongly continuous, i.e. the functions $P(\nu)x$ and $H(\nu)x$ for each x are continuous with respect to ν. Therefore the linear operator $Q(\nu)$ is strongly continuous, and so is $Q^{-1}(\nu)$.

Let us consider now the operator

$$\tilde{U}(\nu, x) = Q^{-1}(\nu)U[\nu, Q(\nu)x].$$

Because of strong continuity of the operators $Q(\nu)$ and $Q^{-1}(\nu)$ the operator $\tilde{U}(\nu, x)$ possesses the same properties of continuity and "smoothness" at zero as $U(\nu, x)$ does. Namely, $U(\nu, x)$ is completely continuous with respect to both the variables and it can be represented in the form

$$\tilde{U}(\nu, x) = \tilde{V}(\nu)x + \tilde{v}(\nu, x),$$

where $\tilde{V}(\nu) = Q^{-1}(\nu)V(\nu)Q(\nu)$ is the linear operator and where $\tilde{v}(\nu, x) = o(\|x\|)$ uniformly with respect to ν. Direct verification shows that for all ν close to ν_0 the linear operator $\tilde{V}(\nu)$ commutes with the projector P and, what is more, satisfies the condition (12). Finally, the sequences $\{x_n\}$ and $\{\nu_n\}$ satisfy the conditions $x_n = U(\nu_n, x_n)$, $\nu_n \to \nu_0$, $x_n \to 0$ ($x_n \neq 0$) if and only if the sequences $\{\tilde{x}_n\}$, $\{\nu_n\}$, where $\tilde{x}_n = Q^{-1}(\nu_n)x_n$, satisfy the same conditions for the operator $\tilde{U}(\nu, x)$: $\tilde{x}_n = \tilde{U}(\nu_n, \tilde{x}_n)$, $\nu_n \to \nu_0$, $\tilde{x}_n \to 0$ ($\tilde{x}_n \neq 0$).

So, without loss of generality one may assume that for the linear part of the operator $U(\nu, x)$ the identities (11) and (12) are valid.

Remark. When outlining in [4] the proof of the theorem we by mistake asserted that the projector $P(\nu)$ and the linear operator $Q(\nu)$ depend on ν continuously with respect to the norm of the operators. Of course, this is not true. The way to correct the mistake has been demonstrated above.

Step 2. Let us set $E_0 = PE$ and $E^0 = (I - P)E$. In what follows it will be convenient to interpret the Bannach space E as the Cartesian product $E = E_0 \times E^0$ of its subspaces E_0 and E^0. In this case without loss of generality one may assume that $\|x\| = \|Px\| + \|(I - P)x\|$.

Let us consider for each n a cylinder $\Omega_n = B_n \times B^n$ in E where

$$B_n = \{x \in E_0 : \|x - q_n g\| \leqslant \tfrac{1}{4} q_n \|g\|\},$$
$$B^n = \{x \in E^0 : \|x\| \leqslant \tfrac{1}{4} q_n \|g\|\}.$$

and where the numbers $q_n > 0$ will be defined later. Without loss of generality one can assume that $\|g\| = 1$, then

$$\tfrac{1}{2} q_n \leqslant \|x\| \leqslant \tfrac{3}{2} q_n \qquad \text{for} \qquad x \in \Omega_n. \tag{14}$$

These inequalities show that Ω_n does not contain the zero point.

The boundary ∂B_n of the ball B_n in the plane E_0 is a Jordan curve. Therefore (see, for example, [9]) for each n there exists a homeomorphism $\nu_n(x)$ of the ball B_n onto the closure D_n of the bounded component of connectedness of the set R^2/L_n.

Let us consider for each n a family of a parameter-depended vector fields

$$\Psi_n(t, x) = R_n(t, x) + S_n(t, x) + T_n(t, x),$$

where $t \in [0, 1]$ and where

$$\begin{aligned} R_n(t, x) &= \{I - V[\nu_n(Px)]\}\{tq_n g + (1 - t)Px\}, \\ S_n(t, x) &= \{I - V[t\nu_0 + (1 - t)\nu_n(Px)]\}(I - P)x, \\ T_n(t, x) &= -(1 - t)v[\nu_n(Px), x]. \end{aligned} \tag{15}$$

We should like to define $q_n > 0$ in such a way that $\Psi_n(t, x) \neq 0$ for all t from the interval $[0, 1]$ and x from the boundary of the cylinder Ω_n. To do this we shall obtain some estimates of the operators R_n, S_n, T_n.

Estimate of $R_n(t, x)$

In virtue of the lemma's conditions the rotation of the vector field $1 - \mu(\nu)$ on L_n is defined. Hence $\mu(\nu) \neq 1$ for $\nu \in L_n$. Then for $\nu \in L_n$ and $x \in E_0$, $x \neq 0$ the following non-equality is valid

$$x - V(\nu)x \neq 0. \tag{16}$$

Indeed, let us suppose the opposite. Then for a certain $\nu \in \mathrm{L}_n$ the operator $V(\nu)$ has the eigenvalue 1 with the eigenvector belonging to E_0. On the other hand the restriction of the operator $V(\nu)$ to its invariant subspace E_0 has exactly two eigenvalues (see (12)), $\mu(\nu)$ and $\bar{\mu}(\nu)$, and these eigenvalues are not equal to 1 for $\nu \in \mathrm{L}_n$. We have come to the contradiction. Consequently (16) is valid.

By virtue of (16) there exists $r_n > 0$ such that

$$\|[I - V(\nu)]Px\| \geqslant r_n \|Px\| \qquad \text{for} \qquad \nu \in L_n. \, x \in E.$$

By definition, $\nu_n(x)$ is a homeomorphism between B_n and D_n. Therefore, $\nu_n(Px) \in L_n$ when $x \in \partial B_n \times B^n$. Consequently

$$\|R_n(t,x)\| = \|\{I - V[\nu_n(Px)]\}\{tq_n g + (1-t)Px\}\|$$
$$\geq r_n \|tq_n g + (1-t)Px\|$$

for $x \in \partial B_n \times B^n$. Here, by definition of B_n, for all $t \in [0, 1]$ the norm $\|tq_n g + (1-t)Px\|$ is greater than $\frac{3}{4}q_n$. Thus

$$\|R_n(t,x)\| \geq \tfrac{3}{4} r_n q_n \qquad \text{for } x \in \partial B_n \times B^n. \tag{17}$$

Estimate of $S_n(t, x)$

Let us show that for a certain $r_* > 0$ and for every ν sufficiently close to ν_0 the following inequality is true

$$\|[I - V(\nu)]x\| \geq r_* \|x\| \qquad \text{for } x \in E^0. \tag{18}$$

Indeed, supposing the contrary one can find $\nu_n \to \nu_0$ and $x_n \in E^0(\|x_n\| = 1)$ for which

$$x_n - V(\nu_n)x_n \to 0. \tag{19}$$

Due to complete continuity of the operator $V(\nu)x$ with respect to both the variables we can assume without loss of generality that $V(\nu_n)x_n \to x_*$. Then from (19) it follows that $x_n \to x_*$ and hence $x_* \in E^0$, $\|x_*\| = 1$. Taking the limit in (19) we obtain

$$x_* = V(\nu_0)x_*.$$

This equality means that in the subspace E^0 the operator $V(\nu_0)$ has an eigenvector corresponding to the eigenvalue 1. The latter contradicts the definition of the subspace E^0. This contradiction proves the inequality (18).

By the lemma's condition $L_n \to \nu_0$, therefore $D_n \to \nu_0$. At the same time $\nu_n(x) \in D_n$ for $x \in B_n$. Hence in virtue of (18) there exists $n_0 > 0$ such that

$$\|\{I - V[t\nu_0 + (1-t)\nu_n(Px)]\}x\| \geq r_* \|x\|$$

for $n \geq n_0$, $x \in E^0$. Then for $n \geq n_0$, $x \in E$

$$\|\{I - V[t\nu_0 + (1-t)\nu_n(Px)]\}(I-P)x\| \geq r_* \|(I-P)x\|.$$

If here $x \in B_n \times \partial B^n$ (where ∂B^n is the boundary of the ball B^n) then $\|(I-P)x\| = \frac{1}{4}q_n$. Hence

$$\|S_n(t,x)\| \geq \tfrac{1}{4} r_* q_n \qquad \text{for } n \geq n_0, x \in B_n \times \partial B^n. \tag{20}$$

Estimate of $T_n(t, x)$

By assumption, $v(\nu, x) = o(\|x\|)$ uniformly with respect to ν from a neighborhood of ν_0. Hence there exists a nondecreasing positive function $c(t)$ $(t > 0)$ such that $c(t) \to 0$ when $t \to 0$ and such that

$$\|v(\nu, x\| \leq \|x\| c(\|x\|).$$

If here $x \in \Omega_n$ then by (14) $\|x\| \leq \frac{3}{2}q_n$. Therefore,

$$\|T_n(t,x)\| \leq \tfrac{3}{2} q_n c(\tfrac{3}{2}q_n) \qquad \text{for } x \in B_n \times B^n. \tag{21}$$

Finally, let us choose q_n. We shall choose $q_n > 0$ in such a way that the following inequalities would be fulfilled

$$R_n(t,x)\| > \|PT_n(t,x)\| \qquad \text{for } t \in [0,1], x \in \partial B_n \times B^n, \tag{22}$$

$$\|S_n(t,x)\| > \|(I-P)T_n(t,x)\| \qquad \text{for } t \in [0,1], x \in B_n \times \partial B^n. \tag{23}$$

Evidently this can be done. Indeed, by virtue of (17), (20) and (21) the inequalities (22) and (23) will be satisfied if the following inequalities are fulfilled

$$\tfrac{3}{4} r_n q_n > \tfrac{3}{2} q_n \|P\| c(\tfrac{3}{2} q_n),$$
$$\tfrac{1}{4} r_* q_n > \tfrac{3}{2} q_n \|I\text{-}P\| c(\tfrac{3}{2} q_n).$$

But the latter inequalities can be satisfied at the cost of a choice of $q_n > 0$. Besides, the sequence $\{q_n\}$ can be chosen in such a way that $q_n \to 0$.

Now, let us show that for q_n chosen in such a manner the vector fields $\Psi_n(t,x)$ do not vanish on the boundary of Ω_n. Indeed, the boundary of cylinder $\Omega_n = B_n \times B^n$ is the union of two sets, $\partial B_n \times B^n$ and $B_n \times \partial B^n$. We shall show that $P\Psi_n(t, Rx) \neq 0$ for $x \in \partial B_n \times B^n$ and $(I-P)\Psi_n(t,x) \neq 0$ for $x \in B_n \times \partial B^n$. Then for all x from the boundary of Ω_n the non-equality $\Psi_n(t,x) \neq 0$ will be fulfilled.

Let $x \in \partial B_n \times B^n$. Since the projector P commutes with the operator $V(\nu)$ (see step 1) then by (15) the operator $P\Psi_n(t,x)$ can be represented in the form

$$P\Psi_n(t,x) = R_n(t,x) + PT_n(t,x).$$

Therefore, by (22)

$$\|P\Psi_n(t,x)\| \geq \|R_n(t,x)\| - \|PT_n(t,x)\| \qquad \text{for } x \in \partial B_n \times B^n.$$

From this it follows that $P\Psi_n(t,x) \neq 0$ for $x \in \partial B_n \times B^n$.

Analogously, from (23) it follows that $(I-P)\Psi_n(t,x) \neq 0$ for $x \in B_n \times \partial B^n$.

So,

$$\Psi_n(t,x) \neq 0 \qquad \text{for } t \in [0,1], \quad x \in \partial\Omega_n. \tag{24}$$

Step 3. Let us prove that the vector field $\Psi_n(0,x) = \Phi_n(x)$ has a singular point $x_n \in \Omega_n$ (i.e. $\Phi_n(x_n) = 0$). Then the following relations would be fulfilled (see (14))

$$\tfrac{1}{2} q_n \leq \|x_n\| \leq \tfrac{3}{2} q_n, \qquad \nu_n = \nu_n(Px_n) \in D_n.$$

By construction of q_n, it follows that $x_n \to 0$, $x_n \neq 0$. Owing to the lemma's conditions L_n tends to ν_0, then D_n also tends to ν_0, and therefore $\nu_n \to \nu_0$. Finally, by definition of Φ_n, the equality $\Phi_n(x_n) = 0$ can be rewritten in the form

$$x_n = U(\nu_n, x_n) \qquad \text{where } \nu_n = \nu_n(Px_n).$$

Thus, the lemma will be proved if $\Psi(0,x) = \Phi_n(x)$ has a singular point in Ω_n. To prove this let us estimate the rotation of the vector field $\Psi_n(0,x)$ on $\partial\Omega_n$. For each n the vector field $\Psi_n(t,x)$ is completely continuous with respect to both the variables t and x (by definition [3], this means that the operator $\Psi_n(t,x) - x$ possesses this property). This follows from the complete continuity with respect to ν and x of the operators $V(\nu)x$ and $v(\nu, x)$ (see (15)).

Then on the grounds of (24) (see, for example, [3, 8])

$$\gamma(\Psi_n(0\ \cdot), \partial\Omega_n) = \gamma(\Psi_n(1,\ \cdot), \partial\Omega_n). \tag{25}$$

By virtue of (15) the vector field $\Psi_n(1, x)$ has the form

$$\Psi_n(1, x) = R_n[1, Px] + S_n[1, (I - P)x]$$

where

$$R_n(1, x) = q_n\{I - V[\nu_n(Px)]\}g, \tag{26}$$

$$S_n(1, x) = \{I - V(\nu_0)\}(I - P)x. \tag{27}$$

Since P commutes with $V(\nu)$ (see Step 1) then $R_n(1, x) \in E_0$, $S_n(1, x) \in E^0$. Therefore, on the grounds of the theorem on direct sum of vector fields (see, for example, [3]) the following equality is true

$$\gamma(\Psi_n(1,\ \cdot), \partial\Omega_n) = \gamma(R_n(1,\ \cdot), \partial B_n) \cdot \gamma(S_n(1,\ \cdot), \partial B^n). \tag{28}$$

Let us estimate the factors in the right-hand side of this equality.

By definition of the projector P the spectrum of the restriction of the linear operator $V(\nu_0)$ to $E^0 = (I - P)E$ does not contain the point 1. Therefore [3],

$$\gamma(S_n(1,\ \cdot), \partial B^n) = \gamma(I - V(\nu_0), \partial B^n) = \pm 1. \tag{29}$$

By virtue of (12) and (26) the operator $R_n(1, x)$ has the form

$$R_n(1, x) = q_n\{1 - \mathrm{Re}\,\mu[\nu_n(Px)]\}g - q_n\{\mathrm{Im}\,\mu[\nu_n(Px)]\}h.$$

From this one can see that the operator $R_n(1,\ \cdot): B_n \to E_0$ can be represented as the superposition

$$R_n(1,\ \cdot) = G_n \circ \{1 - \mu(\cdot)\} \circ \nu_n$$

of the operators $\nu_n : B_n \to D_n$, $1 - \mu(\cdot) : D_n \to C^1$ (C^1 is the complex plane) and $G_n : C^1 \to E_0$ where $G_n(\xi + i\zeta) = q_n\xi g + q_n\zeta h$. Since the first and the third operators are homeomorphisms then [3]

$$\gamma(R_n(1,\ \cdot), \partial B_n) = \pm\, \gamma(1 - \mu(\cdot), L_n). \tag{30}$$

By virtue of the lemma's conditions the right-hand side of the latter equality differs from zero.

Now, substituting (29) and (30) in (28) and afterwards substituting (28) in (25) we obtain

$$\gamma(\Psi_n(0,\ \cdot), \partial\Omega_n) \neq 0.$$

As is well known (see, for example, [3, 8]) it follows from the latter non-equality that the vector field $\Psi_n(0, x) = \Phi_n(x)$ takes the zero value in the region Ω_n.

The lemma is proved.

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